Math 217 Fall 2025

Quiz 9 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose V and W are vector spaces. A *linear transformation* $T : V \to W$ is ...

   Solution: A function $T : V \to W$ satisfying

   $$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(\alpha v) = \alpha\, T(v)$$

   for all $u, v \in V$ and all scalars α in the underlying field (here $\mathbb{R}$). Equivalently, $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$ for all u, v and all scalars α, β.

   (b) Suppose V and W are vector spaces and $T : V \to W$ is a linear transformation.

   (i) The *image* of T is ...

   Solution: The subset of the target consisting of all outputs of T:

   $$\operatorname{Im}(T) = \{\, w \in W \mid \exists\, v \in V \text{ with } w = T(v) \,\}.$$

   It is a subspace of W.

   (ii) The *kernel* of T is ...

   Solution: The set of all vectors in the source that T sends to the zero vector of W:

   $$\ker(T) = \{\, v \in V \mid T(v) = 0_W \,\}.$$

   It is a subspace of V.

   (a) Suppose U is a vector space and $u_1, \ldots, u_n \in U$. The *span* of $(u_1, \ldots, u_n)$ is ...

   Solution: The set of all finite linear combinations of these vectors:

   $$\operatorname{span}\{u_1, \ldots, u_n\} = \left\{ \sum_{i=1}^{n} \alpha_i u_i \;\middle|\; \alpha_1, \ldots, \alpha_n \in \mathbb{R} \right\}.$$

   It is the smallest subspace of U containing all u_i.

2. Suppose $n \in \mathbb{Z}_{>0}$. Recall that $\mathcal{P}_n$ denotes the polynomials of degree $\leq n$. Show

   [VS-5:] For all $a \in \mathbb{R}$ and $p(x), q(x) \in \mathcal{P}_n$, $\quad a\,(p(x) + q(x)) = a\, p(x) + a\, q(x)$.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Solution: Write

$$p(x) = \sum_{i=0}^{n} p_i x^i, \qquad q(x) = \sum_{i=0}^{n} q_i x^i.$$

Then

$$p(x) + q(x) = \sum_{i=0}^{n} (p_i + q_i) x^i.$$

By the definition of scalar multiplication in $\mathcal{P}_n$ and distributivity in $\mathbb{R}$,

$$a\,(p(x)+q(x)) = \sum_{i=0}^{n} a(p_i+q_i)x^i = \sum_{i=0}^{n} (ap_i+aq_i)x^i = \sum_{i=0}^{n} (ap_i)x^i + \sum_{i=0}^{n} (aq_i)x^i = (ap)(x)+(aq)(x).$$

Hence $a(p+q) = ap + aq$ in $\mathcal{P}_n$.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) Suppose X and Y are sets. The function $f : X \to Y$ is surjective if and only if its image is equal to its target.

Solution: TRUE. By definition, f is surjective (onto) precisely when for every $y \in Y$ there exists $x \in X$ with $f(x) = y$, i.e., when $\mathrm{Im}(f) = \{f(x) \mid x \in X\} = Y$.

(b) The trace map $\mathrm{tr} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ sending $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $a + d$ is linear and has trivial kernel (that is, its kernel is $\{\vec{0}\}$).

Solution: FALSE. Linearity holds since $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and $\mathrm{tr}(\alpha A) = \alpha\,\mathrm{tr}(A)$. However, the kernel is $\{A \in \mathbb{R}^{2\times 2} \mid \mathrm{tr}(A) = 0\}$, which is not just $\{0\}$. For example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq \mathbf{0} \quad \text{but} \quad \mathrm{tr}(A) = 1 + (-1) = 0.$$

Thus the kernel is nontrivial (a 3-dimensional subspace), so the statement is false.